NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

VOL. XVII BATON ROUGE, LA., October, 1942 No. 1

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VOL. XVII BATON ROUGE, LA., October, 1942 No. 1

Entered as second-class matter at Baton Rouge, Louisiana.

Published monthly excepting June, July, August, and September, by S. T. SANDERS.

The publication of Volume XVII is aided by an appropriation from the Mathematical Association of America.

THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

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2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the MAGAZINE.

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The Magazine to Receive Aid From M. A. of A.

In many issues of this journal we have commented editorially on the solidarity of American mathematical interests. No more satisfying evidence of such solidarity could be desired than that furnished in the letter quoted below from W. D. Cairns, Secretary-Treasurer of the Mathematical Association of America.

Office of the Secretary-Treasurer
97 Elm Street, Oberlin, Ohio

September 14, 1942.

Dear Professor Sanders:

At the meeting of the Board last week, and on the recommendation of our Executive Committee, it was voted that an appropriation of $400 be made in 1943 toward the expense of printing and distributing the volume of the National Mathematics Magazine for 1942-43. This had not come to us as a request from you, but the Board, sensing that this is a special critical year for the journal, felt that we should make this appropriation. I have no doubt that this will be rather pleasing to you and to your associates. As indicated, this is an item in our budget for 1943 and can be paid at whatever time in 1943 you may wish it.

We hope very much, indeed, that, while this gift will, of course, aid materially in the plans for your next volume, the fact that it has been given by the Mathematical Association will inspire others to help in your very fine enterprise.

Yours very truly,
W. D. Cairns,
Secretary-Treasurer.

Our editorial colleagues, subscribers to the Magazine, and its friends will rejoice over this generous and broad-visioned assistance from an organization whose zeal and intelligence in publicizing the values of mathematics have never been excelled, rarely equalled.

The sum which the Association is pledging to aid in "printing and distributing the volume for 1942-43" will go far in assisting us to meet operating expenses for the current year. In addition to the financial value of the donation will be its inspiration for those promoting the Magazine. The faith of the general mathematical public in the mission of National Mathematics Magazine will be deepened.

S. T. Sanders.

Remarks on a Method for Solving the Torsion Problem

By THOMAS JAMES HIGGINS
Tulane University

Although Saint-Venant's torsion problem—to determine the stress distribution in a torsed prism of uniform cross-section—has long enjoyed favor among applied mathematicians, A. C. Stevenson[1] has but recently demonstrated a very powerful, yet simple procedure that readily yields the solution of a selected member of a certain class of cross-sections: *if* the boundary equation of the cross-section can be written as a summation, usually finite,

$$z\bar{z} = A_0 + \tfrac{1}{2}\sum (A_n z^n + \bar{A}_n \bar{z}^n) \qquad |n| \text{ integral}, \geq 1;\ z = x + iy \tag{1}$$

the complex torsion function is immediately discernible as

$$\Omega = \phi(x,y) + i\psi(x,y) = (i/2)\sum A_n z^n \ ; \tag{2}$$

whence (2) yields for the torsion function

$$\phi = (1/2)[(\phi + i\psi) + (\phi - i\psi)] = (i/4)\sum (A_n z^n - \bar{A}_n \bar{z}^n) \tag{3}$$

and for the conjugate torsion function

$$\psi = (-i/2)[(\phi + i\psi) - (\phi - i\psi)] = (1/4)\sum (A_n z^n + \bar{A}_n \bar{z}^n). \tag{4}$$

Necessarily, if the origin of coordinates is within the cross-section n is positive; for as $\phi(0,0)$ is proportional to the longitudinal displacement of the origin, a singularity of ϕ at the origin is physically inadmissible. If, however, the origin is without the cross-section both positive and negative values of n are admissible; commonly the largest positive n and the smallest negative n are not equal.

Now, as the italicized *if* indicates, not all cross-sections can be solved in this fashion; however, the class that can be solved thus is very large and is especially interesting: for it includes many cross-sections bounded by arcs of several different algebraic curves, cross-sections that are not easily solved in other fashion. As the boundary equations of these cross-sections usually are originally known in terms of the real variables x and y, it is desirable to have means whereby inspection of a given boundary equation will reveal whether or not it *can* be written in the required form (1). It is a simple task to develop the desired criteria.

Expanding each of the terms of (1), adding the resulting expressions, and transferring the constant term to the left-hand member we have that:

If the A_n are all real, $A_n = a_n$, (1) must be capable of exhibition in the form

$$(6)\quad x^2+y^2-a_0=a_1x+a_2(x^2-y^2)+a_3(x^3-3xy^2)+a_4(x^4-6x^2y^2+y^4)+\cdots$$
$$\cdots+a_n(\;)+a_{-1}x(x^2+y^2)^{-1}+a_{-2}(x^2-y^2)(x^2+y^2)^{-2}+\cdots$$

where if n is even

$$a_n(\;)=a_n(x^n-{}_nC_2x^{n-2}y^2+{}_nC_4x^{n-4}y^4-\cdots+i^n{}_nC_ny^n),$$

or if n is odd

$$a_n(\;)=a_n(x^n-{}_nC_2x^{n-2}y^2+{}_nC_4x^{n-4}y^4-\cdots+i^{n-1}{}_nC_{n-1}xy^{n-1});$$

If the A_n are all pure imaginary, $A_n = ia'_n$, then

$$(7)\quad x^2+y^2=-a'_1y-a'_2(2xy)-a_3'(3x^2y-y^3)-a_4'(4x^3y-4xy^3)-\cdots$$
$$\cdots-a_n'(\;)-a'_{-1}y(x^2+y^2)^{-1}-a'_{-2}(2xy)(x+y)^{-2}-\cdots$$

where if n is even

$$a_n'(\;)=a_n'({}_nC_1x^{n-1}y-{}_nC_3x^{n-3}y^3+{}_nC_5x^{n-5}y^5-\cdots-i_n{}^nC_{n-1}xy^{n-1}),$$

or if n is odd

$$a'(\;)=a'({}_nC_1x^{n-1}y-{}_nC_3x^{n-3}y^3+{}_nC_5x^{n-5}y^5-\cdots+i^{n-1}{}_nC_ny^n);$$

If A_n is complex, $A_n = a_n + ia_n'$, then

$$x^2+y^2-a_0=\sum[\,(a_n+ia_n')z^n+(a_n-ia_n')\bar{z}^n]$$
$$=\sum[\,(a_nz^n+a_n\bar{z}^n)+(ia_n'z^n-ia_n'\bar{z}^n)\,]$$
$$(8)\qquad =\text{sum of the right-hand members of (6) and (7).}$$

Accordingly, whether or not a given cross-section is solvable in the manner discussed can be discerned immediately by comparing the boundary equation referred to an origin without or within the cross-section with the proper choice of (6), (7), or (8).

But (6), (7), and (8) have value other than as necessary and sufficient conditions: they give at once a form of the conjugate torsion function; and from this latter follow the stress function $\Psi=\psi-\frac{1}{2}(x^2+y^2)$, the lines of shearing stress, the shearing stresses, and the twisting moment. For if (1) is written in the form

$$(9)\qquad z\bar{z}/2-A_0/2=(1/4)\sum(A_nz^n+\bar{A}_n\bar{z}^n),$$

the right-hand side is identical with ψ by virtue of satisfying the equations defining ψ: by (9) it satisfies the boundary condition

$$\psi=\tfrac{1}{2}(x^2+y^2)+\text{an arbitrary constant}; \tag{10}$$

by its form, the sum of a function of a complex variable and the same function of the conjugate of the variable, it is a solution of

$$\nabla^2\psi=0 \quad \text{over the cross-section.} \tag{11}$$

Accordingly, for a given cross-section and to an arbitrary constant the conjugate torsion function is identical with one-half of the right-hand member of (6), (7), or (8) as the case may be.

As a simple illustration of the use of (6), (7), and (8) consider the equilateral triangle with the boundary equation

$$(x-h)[(x+2h)^2-3y^2]=0. \tag{12}$$

Expanding (12) reveals that it can be written in the form of (6); thus

$$x^2+y^2-(4/3)h^2=-(1/3h)(x^3-3xy^2). \tag{13}$$

Accordingly, the general equilateral triangle can be solved by Stevenson's method.

A cross-section not hitherto treated in the torsion literature is that with boundary equation

$$[\sqrt{2}(x^2-y^2)+(x^2+y^2)-a^2][-\sqrt{2}(x^2-y^2)+(x^2+y^2)-b^2]=0; \tag{4}$$

the curvilinear sides of the cross-section are arcs of intersecting hyperbolas. Expanding and writing (14) in the form of (6)

$$\begin{aligned} x^2+y^2-(a^2b^2)/(a^2+b^2) \\ =(a^2+b^2)^{-1}[\sqrt{2}(a^2-b^2)(x^2-y^2)-(x^4-6x^2y^2+y^4)]. \end{aligned} \tag{15}$$

From (15) we obtain for the conjugate torsion function

$$\psi=\tfrac{1}{2}(a^2+b^2)^{-1}[\sqrt{2}(a^2-b^2)(x^2-y^2)-(x^4-6x^2y^2+y^4)]. \tag{16}$$

If $a=b$ the hyperbolas have foci equally distant from the origin and (16) takes the form

$$\psi=-(x^4-6x^2y^2+y^4)/4a^2. \tag{17}$$

Another interesting application of the test applications: we have found that the general equilateral triangle can be solved by Stevenson's method; on the other hand, quite intricate methods have been employed to obtain *explicit* solutions for the isosceles right triangle, the 30°-60°-90° triangle, and the general right and general isosceles

triangles—respectively, *first* solved by Leibenson,[2] Seth,[3] and Kondô.[4] Now, can the solutions of these or other triangles be obtained by Stevenson's method?

If so, the parameters in the equation defining the boundary of the general triangle

$$(y-mx-b)(y-m'x-b')(y-m''x-b'')=0 \tag{18}$$

allow (18) to be exhibited in the form of (6), (7), or (8). Expanding and collecting terms reveals—the algebra is too lengthy to exhibit here—that this is only possible if, typically, $m'=(m\mp\sqrt{3})/(1\pm\sqrt{3}m)$ and $m''=(m\pm\sqrt{3})/(1\mp\sqrt{3}m)$; or the equilateral triangle alone can be so solved.

As a final example: What cross-sections bounded by arcs of two intersecting circles can be solved by Stevenson's method? The general boundary equation can be written in the form

$$[x^2+y^2-r^2][(x-a)^2+y^2-R^2]=0. \tag{19}$$

Rewriting,

$$\begin{aligned} x^2+y^2+(a^2-r^2-R^2) \\ =(a^2r^2-r^2R^2)(x^2+y^2)^{-1}-2ar^2x(x^2+y^2)^{-1}+2ax. \end{aligned} \tag{20}$$

Inspecting (20) we find that it can be written in the form of (6) if: $r=0$, hence but one circle; $a=0$, the only physically admissible case then requiring that $r=R$ or we have concentric circles of equal radii and a common center; $a=R$, one circle has its center on the perimeter of the other. In this third case a negative power of (x^2+y^2) appears in the resulting form of (20); consequently the origin of the cross-section must lie without the cross-section. Accordingly, the cross-section, first studied by C. Weber,[5] is that of a circle of radius R notched by a second circle of radius r, its center contained in the perimeter of the circle of radius R.

In corroboration: the lenticular cross-section has been studied by Wigglesworth and Stevenson,[6] the circle notched by another with its center not on the first by Gronwall[7] and by Shepherd;[8] as our result portends, their analysis is intricate.

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Expert writers on such technical or educational mathematics as is within the scope of programs preparing applicants for naval or air service with the armed forces of the United States are invited to submit their papers on such subject matter for publication in this journal.

—The Editor-and-Manager.

DENVER, PA., September 15, 1942.

DEAR MR. SANDERS:

... Instead of my paying several years subscription in advance for myself, you suggested that I might accomplish the same purpose for the NATIONAL MATHEMATICS MAGAZINE if I would donate several subscriptions to some friends. I felt out several, whom I thought of, about their knowledge of the magazine, and they all intended to subscribe as a result of the mail they received about the magazine earlier in the summer. So I will have to follow my original thought and enclose a check for $6.00 to cover 3 years subscription in advance of my present expiration date.

. . . .

I realize that my $6.00 advance subscription is not a big aid, but I am not one of those benefitting by higher wages as per the current vogue. However, I will keep the NATIONAL MATHEMATICS MAGAZINE in mind to the end of this year and if I feel I can make an outright gift before the end of the year I will, for I WANT TO SEE IT KEPT GOING.

Sincerely,

CLIFFORD MARBURGER.

Some Analogs of the Triangle Geometry in the Kasner Plane

By LOONG CHI-HO
South West Associated University, Kunming, China

1. *Introduction.* In a recent paper* John de Cicco gave an analog of the nine-point circle in the Kasner plane. The object of this note is to add some results of the triangle geometry to what he has obtained. For fundamental definitions, we refer the reader to the paper of John de Cicco. Our theorems belong not only to the Kasner metric group G_3 but also to the Kasner similitude group G_4: $X=ax+h$, $Y=by+k$. Instead of the *standard position*† of a general triangle, we shall take a general triangle in its general position; that is, the general triangle whose vertices are $A_i(x_i, y_i)$, $i=1, 2, 3$; $x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_1$.

Furthermore, we shall take the unique circumscribing parabolic-circle of the general trihorn $A_1A_2A_3$ as the *unit parabolic-circle*; that is, a parabolic circle with center at origin and with radius equal to unity (the canonical form of a parabolic-circle under the Kasner similitude group G_4). The equation of this parabolic-circle is

$$x^2=y. \tag{1}$$

Hence every point on it has its second coordinates y equal to the square of its first coordinate x.

We employ the following notations for the elementary symmetric functions of x_i:

$$s=x_1+x_2+x_3, \quad t=x_2x_3+x_3x_1+x_1x_2, \quad u=x_1x_2x_3. \tag{2}$$

2. *The Orthocenter.* Let $A_iH_i (i=1, 2, 3)$ be an oblique drawn through A_i, making a dihorn angle α with the side A_jA_k. From the very definition of dihorn angle, we can easily get the equations of the obliques

$$A_iH_i : y=\alpha(s-x_i)x+x_i^2-\alpha(s-x_i)x_i. \tag{3}$$

*John de Cicco, *An Analog of the Nine-Point Circle in the Kasner Plane*, American Mathematical Monthly, Vol. 46 (1939), pp. 627-634.

†John de Cicco took as the standard position the general triangle whose vertices are (0, 0), (1, c), (a,b).

Now A_iH_i are concurrent if and only if

$$\begin{vmatrix} \alpha(s-x_1) & 1 & x_1^2-\alpha(s-x_1)x_1 \\ \alpha(s-x_2) & 1 & x_2^2-\alpha(s-x_2)x_2 \\ \alpha(s-x_3) & 1 & x_3^2-\alpha(s-x_3)x_3 \end{vmatrix} = \alpha(\alpha+1)(x_2-x_3)(x_3-x_1)(x_1-x_2)=0. \tag{4}$$

Now $x_2\neq x_3,\ x_3\neq x_1,\ x_1\neq x_2$. $\alpha=0$ gives three parallel lines. Hence we have

Theorem 1. *The obliques drawn from each vertex of a general trihorn are concurrent if and only if each of the obliques makes a dihorn angle* -1 *with the sides of the trihorn.*

The point of concurrency is an invariant under the group G_3. It is an analog of the orthocenter, and will be referred to as *the horn center* of the general trihorn. Its coordinates are found to be $(-s, s^2)$. Obviously, these coordinates satisfy the equation (1). Hence we have

Theorem 2. *The horn center of a general trihorn lies on the circumscribing parabolic-circle of the trihorn.*

Since, for instance,

$$-(-s+x_2+x_3)=x_1, \quad (-s+x_2+x_3)^2=x_1^2,$$

we obtain

Theorem 3. *If one of four points on a parabolic-circle is the horn center of the other three, then any one of them is the horn center of the remaining three.*

3. *The Orthopole.* Let us determine an analog of the orthopole of a general line

$$\Delta : y=px+r. \qquad p\neq 0,\ \infty, \tag{5}$$

with respect to a general trihorn $A_1A_2A_3$.

Let D_i $(i=1, 2, 3)$ be the foot of the oblique to Δ through A_i, making a dihorn angle α with Δ. Then

$$A_iD_i : y=\alpha px+x_i^2-\alpha px_i. \tag{6}$$

Solving (5) and (6) we obtain the coordinates of D_i,

$$\xi_i=\frac{x_i^2-\alpha px_i-r}{p(1-\alpha)} \qquad \eta_i=\frac{x_i^2-\alpha px_i-\alpha r}{1-\alpha}. \tag{7}$$

Let E_i be the foot of the oblique to A_jA_k through D_i, making a dihorn angle β with A_jA_k. Then

$$D_iE_i : y=\beta(s-x_i)x+\eta_i-\beta(s-x_i)\xi_i. \tag{8}$$

These three lines are concurrent if and only if

$$\begin{vmatrix} \beta(s-x_1) & 1 & \eta_1-\beta(s-x_1)\xi_1 \\ \beta(s-x_2) & 1 & \eta_2-\beta(s-x_2)\xi_2 \\ \beta(s-x_3) & 1 & \eta_3-\beta(s-x_3)\xi_3 \end{vmatrix} = \frac{\beta(1-\alpha\beta)(x_2-x_3)(x_3-x_1)(x_1-x_2)}{1-\alpha}=0.$$

Now $x_2 \neq x_3$, $x_3 \neq x_1$, $x_1 \neq x_2$. $\beta=0$ gives three parallel lines. Hence

Theorem 4. *If an oblique is drawn from each of the vertices of a general trihorn, making a dihorn angle α with a line Δ, then the obliques through the feet of the obliques on Δ, making a dihorn angle β with the opposite sides are concurrent if and only if*

$$\alpha\beta=1. \tag{9}$$

If (9) is satisfied, the point of concurrency is found to be

$$\omega : \begin{cases} x=\dfrac{\alpha p^2-sp+\beta r+\beta t}{(1-\beta)p}, \\ y=\dfrac{-\alpha sp^2-\alpha pr+s^2p-\beta(st-u)}{(1-\alpha)p}. \end{cases} \tag{10}$$

The point ω is an invariant under the group G_3. It is an analog of the orthopole of Δ with respect to the trihorn $A_1A_2A_3$, and will be called *generalised horn-pole with dihorn angle* α of Δ with respect to the general trihorn $A_1A_2A_3$.

Now let us have four points $A_i(x_i, x_i^2)$, $i=1, 2, 3, 4$, on the *fundamental parabolic-circle* (1). Here let us employ the following notations:

$$\begin{aligned} S&=x_1+x_2+x_3+x_4, \\ T&=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4, \\ U&=x_2x_3x_4+x_1x_3x_4+x_1x_2x_4+x_1x_2x_3, \\ V&=x_1x_2x_3x_4. \end{aligned}$$

Furthermore let s_i, t_i, u_i be equal to, respectively, S, T, U minus all terms containing x_i as a factor.

The equation of the line joining the generalised horn-poles of Δ with respect to trihorns, for instance, $A_2A_3A_4$ and $A_1A_3A_4$ is

$$\begin{vmatrix} 1 & (1-\beta)px & (1-\alpha)py \\ 1 & \alpha p^2-s_1p+\beta r+\beta t_1 & -\alpha s_1p^2-\alpha pr+s_1^2p-\beta(s_1t_1-u_1) \\ 1 & \alpha p^2-s_2p+\beta r+\beta t_2 & -\alpha s_2p^2-\alpha pr+s_2^2p-\beta(s_2t_2-u_2) \end{vmatrix}=0, \quad \text{or}$$

$$p(1-\alpha)[(p-\beta S)x+y] = -\alpha^2p^3+\alpha Sp^2-(1+\alpha)pr-Tp+\beta Sr+\beta U. \tag{11}$$

The coefficients of this equation are symmetric functions of x_i, hence

Theorem 5. *The generalised horn-poles of a line with respect to the four trihorns formed by taking each tripet out of four points on the fundamental parabolic-circle* (1) *are collinear.*

4. *The Simson Line.* We shall now consider an analog of the Simson line with respect to a general trihorn. Let $P(X, Y)$ be a point in the Kasner plane, D_i be the foot of the oblique drawn through P to the sides A_jA_k, making a dihorn angle α with A_jA_k. Then the coordinates of D_i are found to be

$$(12)\qquad D_i:\left\{\begin{aligned}\xi_i&=\frac{-\alpha(s-x_i)X+Y+x_jx_k}{(1-\alpha)(s-x_i)},\\ \eta_i&=\frac{-\alpha(s-x_i)X+Y+\alpha x_jx_k}{1-\alpha}.\end{aligned}\right.$$

These three points are collinear if and only if

$$\begin{vmatrix}1&\xi_1&\eta_1\\1&\xi_2&\eta_2\\1&\xi_3&\eta_3\end{vmatrix}=-\frac{\alpha(x_2-x_3)(x_3-x_1)(x_1-x_2)}{(1-\alpha)^2(st-u)}(XY-sY+tX-u)=0.$$

Now $x_2\neq x_3$, $x_3\neq x_1$, $x_1\neq x_2$. $\alpha=0$ gives three parallel lines, hence

Theorem 6. *The feet of the obliques drawn through a point P, making a dihorn angle α with the sides A_jA_k are collinear if and only if P lies on the conic*

$$(13)\qquad XY-sY+tX-u=0.$$

The conic (13) is an equilateral hyperbola with center at $(s, -t)$. Substituting $Y=X^2$ in this equation we get

$$X^3-sX^2+tX-u=0.$$

This shows that the hyperbola (13) circumscribes the trihorn $A_1A_2A_3$. Moreover, one of the asymtotes of the hyperbola (13) is a zero line, while the other is an infinite line.

When P lies on the hyperbola (13), the equation of the line of collinearity is found to be

$$(14)\qquad \begin{aligned}&(1-\alpha)(X-s)[\alpha(-X+s)x-y]\\&\qquad=\alpha^2X^3+\alpha(1-2\alpha)sX^2+(t-s^2\alpha+s^2\alpha^2)X-u.\end{aligned}$$

We shall call this line as the generalised Simson line of P with respect to the general trihorn $A_1A_2A_3$, with dihorn angle α.

In particular, if the dihorn angle $\alpha = -1$, the equation (14) becomes

$$2(X-s)[(X-s)x-y] = X^3 - 3sX^2 + (2s+t)X - u. \tag{15}$$

It can be easily verified that this equation is satisfied by the coordinates of the midpoint of the segment joining P to the horn center. Hence

Theorem 7. *The segment joining a point P on the hyperbola* (13) *to the horn center is bisected by the Simson line of P.*

Let a line $\Delta : y = px + r$ cut the hyperbola (13) in $P_1(X_1, X_1^2)$, $P_2(X_2, X_2^2)$, then we find

$$X_1 + X_2 = \frac{sp - r - t}{p}, \qquad X_1X_2 = \frac{-sr-u}{p}. \tag{16}$$

The equations of the generalised Simson lines of P_1, P_2 with the dihorn angle α are

$$(1-\alpha)[-\alpha(X_1-s)^2x - (X_1-s)y]$$
$$= \alpha^2X_1^3 + \alpha(1-2\alpha)sX_1^2 + (t - s^2\alpha + s^2\alpha^2)X_1 - u,$$
$$(1-\alpha)[-\alpha(X_2-s)^2x - (X_2-s)y]$$
$$= \alpha^2X_2^3 + \alpha(1-2\alpha)sX_2^2 + (t - s^2\alpha + s^2\alpha^2)X_2 - u,$$

and their point of intersection is found to be

$$x = \frac{p^2 - \alpha sp + \alpha^2 r + \alpha^2 t}{\alpha(1-\alpha)p}, \qquad y = \frac{sp^2 + pr - \alpha s^2 p + \alpha^2 st - \alpha^2 u}{(1-\alpha)p}.$$

Comparing these with (10), we get

Theorem 8. *The generalised Simson lines with dihorn angle α of the intersections of a line Δ with the hyperbola* (13) *intersect at the generalised horn-pole of Δ with dihorn angle $1/\alpha$.*

A very large number of our subscribers whose subscriptions expire with the October issue have already sent in their renewal checks. Such thoughtfulness helps greatly in the work of administering the Magazine from the Baton Rouge office, because it makes unnecessary the making out and mailing of formal statements to the subscribers. This kind of cooperation is important at the present time when all administrative details of the Baton Rouge office, secretarial and others, must be handled by the Editor-and-Manager.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

A Fourth Lesson in the History of Mathematics

By G. A. MILLER
University of Illinois

10. *Greek Contributions.* The period of very active contributions of the ancient Greeks to the advance of pure mathematics was very brief, extending only through about two centuries, from the time of Plato and Eudoxus (about 400 B. C.) It is most clearly represented by the *Elements* of Euclid, written about 325 B. C. in the Greek language and still widely regarded as the most influential mathematical work now extant. It has not been preserved in its original form but it has been transcribed and retranscribed many times and these transcriptions differ in many points. Hence there is some doubt in regard to the exact form in which Euclid wrote it and this has given rise to an extensive historical literature. The influence of this pure mathematics on the general standard of mathematics in antiquity has been great but it has been unduly emphasized by various writers, as is usually the case in the history of our subject.

Euclid's *Elements* have been translated into many different languages and it is questionable whether any other work, except the Bible, has passed through as many different editions as these *Elements*, notwithstanding the fact that it is a theoretical work on a somewhat special kind of problems, rather than a practical one from the modern point of view. The recent discoveries relating to ancient Babylonian and ancient Egyptian mathematics have made it clear that the early Greek mathematicians were more largely indebted to earlier developments than has been commonly assumed formerly. This was partly due to the fact that Euclid's *Elements* contain no historical notes and hence it is frequently difficult to determine what Euclid and his Greek predecessors added to the mathematics which was transmitted to them.

A great part of the contents of Euclid's *Elements* is embodied in our modern courses in elementary mathematics. In geometry the modern presentations frequently do not differ widely from those found in Euclid but in algebra there is usually a very wide difference between the modern treatments and those which appear in Euclid's *Elements* and relate to the same subjects. This is largely due to the fact that the ancient Greeks did not use negative and complex numbers. It is of great historical interest to note how greatly the treatment of algebraic subjects has been simplified by the modern general use of these numbers and how much Greek mathematics was disfigured as a result of the failure to extend the number concept by the Greek writers on our subject.

In ancient Greek mathematics the theory of proportion largely took the place of the theory of equations in the modern textbooks and it is interesting to note that many of the works on algebra even in the seventeenth and eighteenth centuries still emphasized proportion unduly. One of the noted algebras of the seventeenth century was published by J. Wallis in the English language in 1685 and was translated into Latin in 1693. In this Latin edition more than fourteen folio pages are devoted to the theory of proportion. This translation into Latin is evidence of the emphasis on the Latin language in England at that time, and the emphasis on proportion which is exhibited in this work is evidence of the slowness with which the theory of proportion was replaced by the theory of equations. In his *Easy Mathematics* (1906), Sir Oliver Lodge said "an equation is the most serious and important thing in mathematics," which is especially interesting historically.

The great influence of the ancient Greeks on the development of mathematics may be inferred from the large number of the common mathematical terms which are derived from the Greek language. Among these are the following: analysis, arithmetic, astronomy, asymptote, axiom, diameter, dynamics, ellipse, geodesy, geometry, harmonic, homogeneous, homologous, hyperbola, isomorphic, logarithm, mathematics, mechanics, method, parabola, parallel, parameter, physics, polygon, polyhedron, polynomial, sphere, statics, stereometry, symmetric, system, and trigonometry. Some of these words were not used by the ancient Greeks but were later named in honor of them. The ancient Greeks had no special name for algebra but they used a kind of geometric algebra, which appears already in Euclid's *Elements*.

These *Elements* stand on a much higher level of mathematical achievement than any earlier extant mathematical work, and the

discoveries of Archimedes (282-212 B. C.) and Apollonius (about 265-170 B. C.) have no equal in pre-Grecian mathematics. It appears now somewhat singular that the ancient Greeks did not give more explicit credit to the work of their predecessors. They were however the first, as far as we now know, to attempt to write a history of mathematics. From the modern point of view they emphasized unduly geometric methods but, contrary to what has been claimed, they did not explain numbers geometrically.* When they discovered the existence of irrational quantities they did not extend their number concept so as to include what we now call irrational numbers. They also never gave more than one root of any of the quadratic equations whose solutions they considered, even when both of the roots are positive.

Among the complete editions of Euclid's *Elements* the one by T. L. Heath, entitled *The Thirteen Books of Euclid's Elements* (1908), in three volumes, is favorably known on account of its very extensive historical notes. It may be desirable to emphasize here that on page 124 of the first volume of this edition there appears the following statement: "There is of course no foundation for the idea, which has often found its way into textbooks, that the object of the postulates is to declare that the only instrument, the use of which is permitted in geometry, are the rule and compass". It may be added here that there is also no foundation for the common statement that Plato asserted that the constructions in geometry should be restricted to the use of these two instruments. He himself sometimes used other instruments in geometrical constructions but he pointed out that it is desirable in such constructions to use only the stated two instruments whenever this is possible.

Neither Euclid's *Elements* nor the conic sections of Apollonius were the earliest works among the ancient Greeks along their respective lines but they were so much better than these earlier works that they replaced them so completely that little is now known about the earlier works on the same subjects. The conic sections of Apollonius seem to contain more work which was then original than Euclid's *Elements* do and the former is still regarded by many as the peak of the mathematical attainments of the ancient Greeks. The Greek mathematical developments during the third and fourth centuries B. C. were so rapid in comparison with their other mathematical advances that they may be regarded as meteoric. Hence this period is of special interest in the history of pure mathematics and the names of Euclid, Archimedes, and Apollonius are outstanding in the brilliant record of Greek mathematical developments.

*Cf *A Survey of Modern Algebra* by Birkhoff and MacLane, page 63, (1941), for an incorrect expression relating to this situation.

The study of arithmetic seems to have preceded that of geometry even among the ancient Greeks, and Pythagoras is said to have asserted that all is number. It has often been assumed that the ancient Babylonians were especially gifted along the line of arithmetic and that the ancient Greeks were especially gifted along the line of geometry, but there seems to be no good reason for such an assumption. People acquire proficiency along the lines which they especially cultivate and this may explain the relatively great advances made by the ancient Greeks along the line of geometry. In modern times there have been many instances of the shifting of mathematical interests among the same people as is illustrated by the emphasis on the group concept during the second half of the nineteenth century, which seems to have been partly due to the simplicity of the postulates upon which an abstract theory of this subject is based and growing emphasis on abstract mathematics. In addition to their own discoveries the ancient Greeks geometrized some of the earlier results which came to them from Babylon and Egypt and thus they created a kind of geometric algebra, as was noted above.

Their algebra was especially weak along the line of the number concept. It has frequently been stated that the ancient Greeks solved the quadratic equations geometrically and that the Arabs later gave a geometric solution of the general cubic equation. In view of the fact that these peoples did not understand complex numbers it is at once evident that they could not have understood the general solutions of these equations in the modern sense of this term, since the theory of complex numbers is just as essential in the general geometric solution of these equations as in their general algebraic solution. One of the most helpful principles in the study of the mathematical contributions of particular early nations is to bear in mind their attainments relating to the number concept. The facts that even negative numbers were not fully understood until about the beginning of the nineteenth century and that complex numbers were not even used properly before the sixteenth century are of profound significance in the history of other subjects.

A very important feature of ancient Greek mathematics is that at least from the time of Plato (429-348 B. C.) to Diophantus (second half of the third century) the concept of number was commonly restricted to positive integers. This was a backward step by Euclid and other Greek mathematicians. Near the beginning of Book VII of Euclid's *Elements* there appears the definition: "A number is a multitude of units." The fact that the ancient Greeks commonly restricted the number concept to discontinuous magnitudes is fundamental in

the history of mathematics. Proposition 7 of Book X of Euclid's *Elements* asserts that incommensurable magnitudes have not to one another the same ratio which a number has to a number. It was not until the time of Diophantus that rational fractions which are not equivalent to integers were commonly regarded as numbers among the Greeks.

The ancient Greeks calculated sometimes with binomials of the form $a-b$, when a and b are positive and a exceeds b. Hence they used such rules as $-\cdot-=+$ and $-\cdot+=-$ in multiplication which had been used by the Babylonians about 400 B. C. They therefore came very close to actual negative numbers, which were employed shortly after Diophantus by the Hindus. *A satisfactory theory of these numbers did however not appear in the mathematical literature until more than a thousand years after the time of Diophantus* but it is interesting to note the occasional proper use of them in special cases long before their legitimate general use was established. The solution of equations was especially in need of a correct theory of negative numbers and the early developments in trigonometry were very much retarded as a result of the lack of correct notions of these numbers, as one can see by trying now to develop the elementary formulas of trigonometry without their use.

The usefulness of negative numbers in elementary trigonometry may be regarded as the last straw which broke the back of the opposition to their general use. Even d'Alembert (1717-1783) contested the multiplication of negative numbers and affirmed, in particular, that a^2-ab+b^2 represents $(a-b)^2$ whenever $a>b$, but $(b-a)^2$ whenever $b>a$. See *Géometres Francais par Niels Nielsen*, 1935, p. 17. Forward steps in mathematics were sometimes due to the observation that their advantages overbalanced their disadvantages. In such cases they naturally encountered opposition before these advantages were fully understood. Hence the history of mathematics is not entirely free from controversial literature. The logarithms of negative numbers led to a large amount of such literature during the greater part of the eighteenth century. This was due to some of the leading mathematicians of that time, including G. W. Leibniz (1646-1716) and John Bernoulli (1667-1748), both of whom made important contributions towards the development of our subject and *could be classed with the fifty leading mathematicians of all times* up to the present.

It may also be noted here that while the operations of multiplication and division coalesced very early, since division is equivalent to multiplication by the inverse of the divisor, as was noted already by the ancient Babylonians, these operations nevertheless were commonly

retained as different operations. Somewhat later, when negative numbers were introduced, the operations of addition and subtraction coalesced, but they too were later retained as different operations. The coalescence of multiplication and division is possibly the oldest instance of the union of two mathematical operations into one and hence is of interest in the history of our subject, especially since these operations nevertheless continued to be commonly regarded as different, and are obviously different as regards the operations with respect to natural numbers. Historically the common use of unit fractions by the ancient Egyptians, and the extensive tables of the reciprocals of numbers by the ancient Babylonians may have been influenced by the coalescence of multiplication and division.

The student of the history of mathematics finds much that is unreliable and contradictory in the writings on Greek mathematics. In particular, much that has been written about Pythagoras and his school is without solid foundation. The quotations from Greek writers are frequently contradictory and hence such quotations cannot always be accepted as final. It is therefore not sufficient in all cases to go to the original sources in the study of Greek mathematics. These sources must also be tested as regards reliability. This is also true in regard to the sources relating to the early mathematics of other countries. It therefore results that the study of secondary sources of information is frequently as important as the study of the original sources, especially for the beginner who may be unfamiliar with national tendencies in ancient times. One cannot be more tolerant of historical errors than of other mathematical errors if one aims to understand this history. In fact, it is usually more difficult to detect the former from the context.

11. *Homogeneity*. The geometrization of mathematics by the ancient Greeks led to a deeper insight into many parts of our subject, but it had also some disadvantages which were not completely removed for about a thousand years when the subject of analytic geometry received more attention. The ancient Greeks commonly regarded an expression of the first degree in one unknown as representing a line while an expression of the second degree in one or two unknowns was commonly regarded as representing an area. It naturally did not seem desirable to them to add lines and areas since they have different dimensions. The terms square and cube show how the geometric ideas enter into the algebraic operations. Even in a quadratic equation we thus seem to combine by addition quantities of very different properties.

The interplay between arithmetic and geometry began very early when square units were first employed to measure areas and

cubical units were first used to measure volumes. These gave rise to numerical relations between geometric figures of different dimensions and emphasized the fundamental importance of the concept of number in geometry. *Some of the connections between algebra and geometry were therefore observed long before the invention of analytic geometry in the seventeenth century by R. Descartes* (1596-1650) *and others.* Even the ancient Babylonians combined by addition magnitudes of different dimensions and thus they made use of the penetration of number and the interplay of algebra and geometry in numerical equations. Some of the earliest equations which have been transmitted to us relate to the areas of rectangles as the product of their two adjacent sides and hence they imply the use of square units of area and numerical relations between lines and areas. In fact, the ancient Babylonians considered some equations whose degrees exceed three and thus they gave evidence of their interest in algebra, but they naturally were not able to give a complete solution even of the general quadratic equation in view of their restricted view of numbers. This view never included even a partial view of complex numbers and was far from satisfactory from the modern point of view as regards negative numbers. The ancient Greeks were similarly limited as regards their concept of numbers but by their geometric methods they did much to clarify many results which they obtained from their predecessors. Unfortunately their mathematics soon became very complex as a result of their geometric limitations.

Even now areas are not always measured by square units as may be seen from the use of acres, nor are volumes now always measured by cubical units as results from the common use of bushels. It is impossible now to trace the selection of many of the units of measure to their origins but it is evident that the choice of these units in such a way that a numerical relation becomes evident between quantities of different dimensions is of great mathematical interest. The emphasis which the ancient Greeks placed on making the terms of an equation homogeneous was reflected in much of the later mathematical work in view of the great influence of Greek mathematical works on the writers on pure mathematics in medieval times. This influence continued in later times but was then no longer slavishly followed.

The emphasis on homogeneity in equations is historically interesting because it illustrates a somewhat common tendency in mathematics to fail to recognize which is the more important of two conflicting ideas. It is clear that we can add directly only quantities of the same kind and that we can therefore not add directly lines, areas and solids, but when the numerical values of all of these are expressed

we can obviously add these numbers. An equation involving terms of different degrees is therefore not meaningless if these terms are connected with plus and minus signs. It is however instructive to consider the reasons why a thoughtful people like the Greeks hesitated to abandon the geometric point of view and replace it by the more abstract number theory point of view in such cases. At any rate it is very important in the history of mathematics to keep in mind the emphasis placed on homogeneity by the ancient Greeks and by some of their followers.

Algebra and geometry have sometimes been called the two feet of mathematics and just as human beings find it desirable to use both of their feet without discriminating under ordinary conditions in favor of either so in mathematics progress has usually been most active when algebra and geometry have both been employed without prejudice to either one. The term analytic geometry expresses this union of two possible points of view but it should not be assumed that it represents the earliest step towards this union. Some of the earliest problems in the extant Babylonian mathematics relate to the application of algebra to problems in geometry. In recent times great stress is frequently laid on the advantages of combined courses in elementary mathematics but it should be emphasized that traces of such a union are as old as the records of mathematics itself. The Greek experiment in trying to geometrize mathematics may be compared to that of a man who when one of his feet was hurt tried to get along with one foot instead of getting the other healed by the extension of the number concept so as to include irrational numbers. It took more than two thousand years to effect this cure completely.

REYNOLDS HIGH SCHOOL,
REYNOLDS, IND., September 21,1942.

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A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from May, 1942, issue)

ANNALS OF MATHEMATICS

Edited by

ORMOND STONE and WILLIAM M. THORNTON.

Office of Publication: University of Virginia.

VOLUME I.

CHARLOTTESVILLE, VA.:

Printed for the Editors by Blakey & Prout, Steam Book and Job Printers

Agents: B. Westermann & Co., *New York.*

Following the publication of the *American Journal of Mathematics* was the founding of the ANNALS OF MATHEMATICS, founded by Ormond Stone. It starts without a word of explanation for its existence, or what it seeks to accomplish, or how it purposes to conduct itself.

However, in Vol. II, following the Title Page and a page of Indices, we have the following enlightening statement: *Annals of Mathematics*, First Series.

The *Annals of Mathematics* was founded by Professor Ormond Stone of the University of Virginia, in 1884, the first number appearing in March of that year.

The ANNALS was the successor of the *Analyst*, a journal of mathematics founded and edited by Dr. Joel E. Hendricks, of Indiana.

Professor Stone supported the publication of the ANNALS OF MATHEMATICS from his private income during ten years, at the end of which time, Professor Stone feeling unable to continue the expense of the publication, that responsibility was assumed by the authorities of the University of Virginia, and its publication was thus continued until the close of the twelfth volume, issued in June, 1899.

On the withdrawal, by the University of Virginia of the appropriations, necessary for its publication, the responsibility for its support was assumed by Harvard University.

The Second Series issued under the auspices of Harvard will begin, with the first number of the first volume of that series, to appear about October, 1899.

As the ANNALS is still being published and, it is hoped, will be, for many years untold in the future, it is readily available to students to be consulted without much difficulty. We shall, therefore, give only a brief account of this important journal.

Vol. I, consists of six numbers of 24 pages each, the printed page being 5 inches by 7 inches. Originally, it was published bi-monthly.

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$$\frac{dy}{dx}+y^2+Dy+Q=0,$$

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The Teachers' Department

Edited by
JOSEPH SEIDLIN, JAMES MCGIFFERT, J. S. GEORGES
and L. J. ADAMS

On the General Definition of a Conic

(A Lesson in Plane Analytic Geometry)

By ALEXANDER W. BOLDYREFF and FRANZ E. HOHN
University of Arizona

I. *Introduction.* It is assumed that the student has derived the equations of the parabola, the ellipse, and the hyperbola, using in each case the optimum position for the axes of reference; has established the focus-directrix property in each case; and has learned the transformation of coördinates by translation and by rotation of axes.

The following treatment has for its purpose to serve as a supplement to his study of the conventional analysis of the loci represented by the general quadratic equation in two variables.

It is believed that the student will thus acquire a clearer and a deeper insight into the problem *as a whole.* At the same time it is believed that the main results are established in an easier and in a more direct way.

II. *Definition.* A *conic* is a locus of the point $P(x,y)$ satisfying the condition:

$$|d_1| = e|d_2|,$$

where d_1 is the distance from P to a fixed point $F(\alpha,\beta)$, the *focus;* d_2 is the distance from P to a fixed straight line $D(ax+by+c=0)$, the *directrix;* and e is a positive constant, the *eccentricity.*

If $e>1$, the conic is an *hyperbola,*

$e=1$, the conic is a *parabola,*

$e<1$, the conic an *ellipse.*

The line A passing through F at right angles to D is defined as the *principal axis* of the conic.

III. *Derivation of the General Equation of a Conic.* From the definition:

$$\sqrt{(x-\alpha)^2+(y-\beta)^2}=e\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|, \text{ or} \tag{1}$$

$$\begin{aligned}(a^2+b^2-a^2e^2)x^2-2abe^2xy+(a^2+b^2-b^2e^2)y^2-2[\alpha(a^2+b^2)+ace^2]x\\ -2[\beta(a^2+b^2)+bce^2]y+(a^2+b^2)(\alpha^2+\beta^2)-c^2e^2=0.\end{aligned} \tag{2}$$

This is of the form:

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0. \tag{3}$$

Thus the general equation of a conic is a quadratic equation in x and y.

Conversely, the locus of (3) is a conic, real or imaginary, or a limiting case of a conic, since it can be shown that the six quantities a, b, c, e, α, and β can be determined to satisfy the six equations:

$$\begin{aligned}a^2+b^2-a^2e^2&=A\\ a^2+b^2-b^2e^2&=C\\ abe^2&=-B\\ \alpha(a^2+b^2)+ace^2&=-D\\ \beta(a^2+b^2)+bce^2&=-E\\ (a^2+b^2)(\alpha^2+\beta^2)-c^2e^2&=F\end{aligned} \tag{4}$$

and, therefore, (3) can be shown to be equivalent to (1).*

IV. *Test for Distinguishing the Conics.* Let $\Delta=B^2-AC$ be defined as the *characteristic* of the equation (3).

From (4):

$$\Delta\equiv B^2-AC\equiv(a^2+b^2)^2(e^2-1). \tag{5}$$

Therefore, the nature of the conic is determined by the sign of Δ:

If $\Delta>0$, then $e>1$, and the locus of (3) is an hyperbola,

$\Delta=0$, then $e=1$, and the locus of (3) is a parabola,

$\Delta<0$, then $e<1$, and the locus of (3) is an ellipse.

V. *Invariance of Δ on Transformation of Coördinates.* From (5) the invariance of Δ depends on the invariance of a^2+b^2, the sum of the squares of the coefficients of x and y in the equation of the directrix.

*Cf. K. P. Williams, NATIONAL MATHEMATICS MAGAZINE, October, 1941, p. 37.

Let $$x = x' \cos \Theta - y' \sin \Theta + h,$$
$$y = y' \cos \Theta + x' \sin \Theta + k\ ;$$

The equation of D becomes:

$$(a \cos \Theta + b \sin \Theta)x' + (b \cos \Theta - a \sin \Theta)y' + ah + bk + c = 0,$$

and since

$$(a \cos \Theta + b \sin \Theta)^2 + (b \cos \Theta - a \sin \Theta)^2 \equiv a^2 + b^2,$$

the invariance of Δ follows immediately.

$A + C$ is also an invariant of the equation (3), for from (4):

$$A + C \equiv (a^2 + b^2)(2 - e^2).$$

VI. *Simplification of the General Equation.* Let the Principal Axis of the conic have an inclination Θ, $0 \leqslant \Theta < \pi$.

If $\Theta = 0$ or $\Theta = \pi/2$, A is parallel to the x-, or to the y-axis, the equation (3) will have no xy-term and is reduced to the simplest form by a translation of axes.

If $0 < \Theta < \pi/2$, then the rotation of the axes of coördinates through the angle Θ will make the principal axis parallel to the x-axis. (Fig. 1).

Finally if $\pi/2 < \Theta < \pi$, the rotation of the coördinate axes through the angle $\Theta - \pi/2$ will make the principal axis parallel to the y-axis. (Fig. 2).

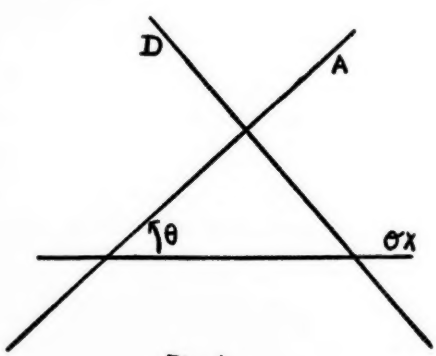

Fig. 1

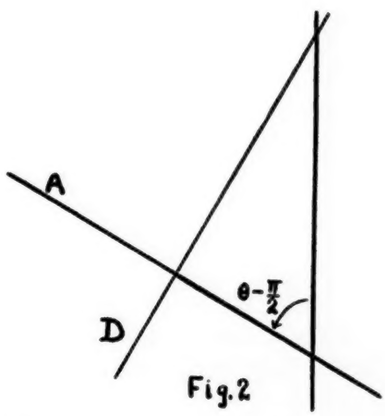

Fig. 2

Thus the last two cases are reducible to the first by rotating the axes through a positive acute angle.

The transformation equations for the required rotation of the axes are easily deduced from (3).

From the first three equations of (4):

$$\frac{a}{b}-\frac{b}{a}=\frac{A-C}{B}\ .$$

But, since A is perpendicular to D: $ax+by+c=0$, $m=\tan\theta=b/a$.

Hence $$\frac{1-m^2}{m}=\frac{A-C}{B}\ ,\quad\text{or}$$

$$\tan(2\theta+n\pi)=\frac{2B}{A-C}\ .$$

It is always possible to find an angle θ', $0<\theta'<\pi/2$, such that

$$\tan 2\theta'=\frac{2B}{A-C}\ . \tag{6}$$

And it is clear that θ' is equal to θ, if θ is acute, and to $\theta-\pi/2$, if θ is obtuse.

From (6), letting $T=\tan 2\theta'$,

$$K=\cos 2\theta'=\frac{T}{|T|\sqrt{T^2+1}}\ ,$$

since K and T must have like signs.

Now, $$\sin\theta'=\sqrt{\frac{1-K}{2}}\ ,\quad \cos\theta'=\sqrt{\frac{1-K}{2}}\ ,$$

and the required transformation equations are:

$$x=x'\sqrt{\frac{1+K}{2}}-y'\sqrt{\frac{1+K}{2}} \tag{7}$$

$$y=y'\sqrt{\frac{1+K}{2}}+x'\sqrt{\frac{1-K}{2}}\ .$$

Algebraically transformation (7) is equivalent to the removal of the xy-term from the equation of the conic. For in the transformed equation the coefficient of $x'y'$ is

$$-2A\sin\theta'\cos\theta'+2C\sin\theta'\cos\theta'+2B(\cos^2\theta'-\sin^2\theta')$$

$$=2B\cos 2\theta'-(A-C)\sin 2\theta'$$

$$=(A-C)\cos 2\theta'\left[\frac{2B}{A-C}-\tan 2\theta'\right]=0.$$

Let the transformed equation of the conic be

$$A'x'^2+C'y'^2+2D'x'+2E'y'+F=0. \tag{8}$$

From (5):

$$\Delta \equiv -A'C' \equiv (a^2+b^2)^2(e^2-1).$$

There are three possibilities:

Case I. The conic is a parabola if $e=1$, i. e. if either A' or C' is zero. A' and C' cannot both be zero, because the invariant

$$A+C \equiv (a^2+b^2)(2-e^2) \equiv (a^2+b^2) \neq 0.$$

Therefore (8) can be written:

$$\left(x'+\frac{D'}{A'}\right)^2 = -\frac{2E'}{A'}\left(y'-\frac{D'^2-A'F}{2A'E'}\right), \text{ if } C'=0, \text{ or}$$

$$\left(y'+\frac{E'}{C'}\right)^2 = -\frac{2D'}{C'}\left(x'-\frac{E'^2-C'F}{2C'D'}\right), \text{ if } A'=0.$$

These reduce respectively to:

$$x''^2=2py'', \text{ where } x''=x'+\frac{D'}{A'},\ y''=y'-\frac{D'^2-A'F}{2A'E'},\ p=-\frac{E'}{A'}, \text{ and}$$

$$y''^2=2px'', \text{ where } y''=y'+\frac{E'}{C'},\ x''=x'-\frac{E'^2-C'F}{2C'D'},\ p=-\frac{D'}{C'}.$$

The discussion of limiting cases is left to the student.

Case II. The conic is an ellipse if $e<1$, i. e. if $A'C'>0$. Without loss of generality assume A' and C' *both* positive. Then (8) can be written:

$$\frac{\left(x'+\frac{D'}{A'}\right)^2}{\frac{D'^2}{A'^2}+\frac{E'^2}{A'C'}-\frac{F}{A'}}+\frac{\left(y'+\frac{E'}{C'}\right)^2}{\frac{D'^2}{A'C'}+\frac{E'^2}{C'^2}-\frac{F}{C'}}=1, \text{ or}$$

$$\frac{x''^2}{H^2}+\frac{y''^2}{G^2}=1, \text{ where}$$

$$x''=x'+\frac{D'}{A'},\quad y''=y'+\frac{E'}{C'},$$

$$H^2 = \frac{D'^2}{A'^2} + \frac{E'^2}{A'C'} - \frac{F}{A'}, \quad G^2 = \frac{D'^2}{A'C'} + \frac{E'^2}{C'^2} - \frac{F}{C'}.$$

The locus is imaginary if $C'D'^2 + A'E'^2 < A'C'F$.

The locus is real if $C'D'^2 + A'E'^2 > A'C'F$.

It is a point if $C'D'^2 + A'E'^2 = A'C'F$, and a circle if $A' = C'$.

Case III. The conic is an hyperbola if $e > 1$, i. e. if $A'C' < 0$, or A' and C' have opposite signs.

Assume $A' > 0$.

Then (8) can be written as

$$\frac{\left(x' + \dfrac{D'}{A'}\right)^2}{\dfrac{D'^2}{A'^2} + \dfrac{E'^2}{A'C'} - \dfrac{F}{A'}} - \frac{\left(y' + \dfrac{E'}{C'}\right)^2}{\dfrac{F}{C'} - \dfrac{D'^2}{A'C'} - \dfrac{E'^2}{C'^2}} = 1,$$

if $C'D'^2 + A'E'^2 < A'C'F$ or

$$\frac{\left(y' + \dfrac{E'}{C'}\right)^2}{\dfrac{E'^2}{C'^2} + \dfrac{D'^2}{A'C'} - \dfrac{F}{C'}} - \frac{\left(x' + \dfrac{D'}{A'}\right)^2}{\dfrac{F}{A'} - \dfrac{D'^2}{A'^2} - \dfrac{E'^2}{A'C'}} = 1,$$

if $C'D'^2 + A'E'^2 > A'C'F$.

These equations reduce to:

$$\frac{x''^2}{H^2} - \frac{y''^2}{G^2} = 1, \quad \text{where}$$

$$x'' = x' + \frac{D'}{A'}, \quad y'' = y' + \frac{E'}{C'},$$

$$H^2 = \frac{D'^2}{A'^2} + \frac{E'^2}{A'C'} - \frac{F}{A'}, \quad G^2 = \frac{F}{C'} - \frac{D'^2}{A'C'} - \frac{E'^2}{C'^2};$$

or

$$\frac{y''^2}{-G^2} - \frac{x''^2}{-H^2} = 1,$$

where y'', x'', G^2, H^2 are as above.

If $C'D'^2 + A'E'^2 = A'C'F$ the locus is a pair of straight lines.

VII. *Problems.* 1. Show that the equation (3) reduces to $(bx-ay)^2+2Dx+2Ey+F=0$, for any parabola. Hint: use (5).

2. By letting
$$c^2=\frac{(a^2+b^2)r^2}{e^2}$$
and allowing e to tend to zero, deduce from (1) the equation of the circle $(x-\alpha)^2+(y-\beta)^2=r^2$.

3. From the invariant $A+C\equiv(a^2+b^2)(2-e^2)$ prove that the equilateral hyperbola is the only conic from whose equation both the x^2 and the y^2 terms can be eliminated.

4. Prove that e satisfies the equation:
$$\Delta e^4-[4\Delta+(A+C)^2]e^2+4\Delta+(A+C)^2=0.$$
What is the nature of the coefficients in this equation?

5. Find e in 4. if $A+C=0$. Compare with 3.

LINCOLN, NEBRASKA, July 30, 1942.

DEAR PROFESSOR SANDERS:

It is with great pleasure that I enclose my subscription for the next year (October, 1942-May, 1943) of the NATIONAL MATHEMATICS MAGAZINE. It is a fine magazine and gives me much pleasure—both articles and the problem section. I shall do my best to spread good reports of the magazine and induce my mathematically interested friends to subscribe.

I do send my best wishes for the continued success of the publication and hope it will continue to reach an ever-increasing group of mathematically inclined people—professional and amateur.

Sincerely yours,
WILLIAM N. HUFF.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, West Point, N. Y.

SOLUTIONS

No. 437. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find a ten-digit number which will equal the square of the sum of the two numbers formed by its first five digits and its last five digits.

Solution by the *Editors*.

Let x and y be the two 5-digit numbers. Then the hypothesis asserts

$$(x+y)^2 = 10^5x+y,$$

which can be put in the form

$$(x+y-5\cdot 10^4)^2 = 25\cdot 10^8 - 99999y = R^2 \tag{1}$$

where R is an integer. Evidently $|R| < 50000$. Furthermore $R^2 \equiv 25\cdot 10^8 \pmod{99999}$ whence

$$R \equiv \pm 5 \pmod 9),\ R \equiv \pm 20 \pmod{41},\ R \equiv \pm 135 \pmod{271}.$$

These are easily combined ("Chinese Remainder Theorem") into

$$R \equiv \pm 27778,\ \pm 32656,\ \pm 45121,\ \pm 49999,\ (\text{mod } 99999).$$

The values of x and y are found from (1), so that the desired numbers appear as

$$6049417284 = (60494+17284)^2 = 77778^2,$$

$$0493817284 = (04938+17284)^2 = 22222^2,$$

$$6832014336 = (68320 + 14336)^2 = 82656^2,$$
$$0300814336 = (03008 + 14336)^2 = 17344^2,$$
$$9048004641 = (90480 + 04641)^2 = 95121^2,$$
$$0023804641 = (00238 + 04641)^2 = 04879^2,$$
$$9999800001 = (99998 + 00001)^2 = 99999^2,$$
$$0000000001 = (00000 + 00001)^2 = 00001^2,$$

from which it may be desired to exclude the alternate numbers as not being strictly ten-digit numbers.

No. 440. Proposed by *H. T. R. Aude,* Colgate University.

A point interior to a square is at the unequal distances, a, b, and c units from three of the vertices of the square.

(1) Determine the conditions on the three numbers a, b and c which allow the existence of only one square, only two squares, of three squares.

(2) Let the numbers a, b, c be restricted to integers. Of all the admissible sets a, b, c in the three cases cited in (1), find in each case the set a, b, c where the sum $a+b+c$ is the least.

(This problem was suggested by Problem No. 407 proposed by the Mathematics Club of Tulane University and is respectfully dedicated to the members of the Club.)

Solution by the *Proposer.*

A point P, (x,y) is assumed to be inside a square $P_1P_2P_3P_4$ of side K. Let the coordinates of P_1, P_2 and P_3 be $(0,K)$, $(0,0)$ and $(K,0)$, respectively. If the distances P_1P, P_2P, P_3P are represented respectively by the numbers a, b, c, then the assumed conditions give the three equations:

$$x^2+y^2-2Ky+K^2-a^2=0, \qquad x^2+y^2-b^2=0,$$
$$x^2+y^2-2Kx+K^2-c^2=0.$$

Elimination of x and y yields the equation

$$(1) \qquad 2K^4-2(a^2+c^2)K^2+(a^2-b^2)^2+(b^2-c^2)^2=0.$$

This equation will have unequal positive solutions* for K^2 if and only if

$$(a^2+c^2)^2-2(a^2-b^2)^2-2(b^2-c^2)^2>0,$$

*$2K^2=a^2+c^2\pm[(a^2+c^2)^2-2(a^2-b^2)^2-2(b^2-c^2)^2]^{1/2}$. But the sign before the radical must be positive because the triangle P_1PP_3 is obtuse-angled at P and thus $2K^2>a^2+c^2$. Hence there exists not more than one value of K for a given selection of a, b, c (in that order). To obtain the twelve solutions mentioned in the solution of No. 407 (this Magazine, November, 1942, p. 106) we must count the six permutations of a, b, c and allow exterior positions for P as well as interior.—ED.

or the equivalent inequality

$$[(a+c)^2-2b^2]\cdot[2b^2-(a-c)^2]>0,$$

which, since a, b, and c are positive numbers, is true only when

$$a+c>b\sqrt{2} \quad\text{and}\quad |a-c|<b\sqrt{2}. \tag{1}$$

When these two conditions are fulfilled there exists a square of side K such that $P_1P=a$, $P_2P=b$, $P_3P=c$.

Other values of K (hence other squares) may result upon permutation of a, b, c. Evidently the interchange of a and c leaves K unchanged, but the interchange of a and b and of b and c lead, as above, to the respective conditions

$$b+c>a\sqrt{2} \quad\text{and}\quad |b-c|<a\sqrt{2}, \tag{2}$$

$$a+b>c\sqrt{2} \quad\text{and}\quad |a-b|<c\sqrt{2}. \tag{3}$$

For any three given numbers a, b, c, there will be no square, one, two or three squares according as none, or one, or two, or all three of the conditions (1), (2), (3) are satisfied. For example the numbers 1, 3, 6, whatever their order, yield no solution. For 1, 4, 6 there exists one square; for 1, 2, 4 there are two; for 6, 7, 8 there are three.

By canvassing the different partitions of $a+b+c=6, 7, 8, \cdots$ into distinct integers a, b, c, it is easy to verify that the sets having the minimum sums for the four cases above are respectively (1, 2, 4), (1, 3, 5), (1, 2, 3), (4, 5, 6).

No. 442. Proposed by *Paul D. Thomas*, Sherburne, N. Y.

Find the equations of the geodesics on a surface whose linear element is given by $ds^2=(u^2-v^2)(v^2du^2+u^2dv^2)/u^2v^2$ and show that if points on the surface correspond to points in the plane $z=0$ through $u=a\cdot\sec(\log x)$, $v=1/y$, the geodesics correspond to the central conics $x^2-2abxy+b^2=0$, where a and b are constants.

Solution by the *Proposer*.

From the given linear element $E=(u^2-v^2)/u^2$, $F=0$, $G=(u^2-v^2)/v^2$ $H=(u^2-v^2)/uv$. With these values the differential parameter

$$\Delta_1 f=(Ef_v^2-2Ff_uf_v+Gf_u^2)/H^2$$

becomes* $\quad \Delta_1 f=v^2f_v^2/(u^2-v^2)+u^2f_u^2/(u^2-v^2).$

Place $\Delta_1 f=1$, which gives $v^2f_v^2+v^2=u^2-u^2f_u^2$.

*Eisenhart, *Differential Geometry*, page 84.

Set both members of this last equation equal to a constant A, and solve respectively for f_u, f_v to find

$f_u=(u^2-A)^{1/2}/u$ and $f_v=(A-v^2)^{1/2}/v$. These values make

$df=f_u du+f_v dv$ integrable, and

$$f=\int(u^2-A)^{1/2}du/u+\int(A-v^2)^{1/2}dv/v$$

$$f_A=-\tfrac{1}{2}\int du/u(u^2-A)^{1/2}+\tfrac{1}{2}\int dv/v(A-v^2)^{1/2}$$

$$=-(1/2A^{1/2})\sec^{-1}(u/A^{1/2})-(1/2A^{1/2})\log\frac{[A^{1/2}+(A-v^2)^{1/2}]}{v}$$

$$\therefore\qquad \sec^{-1}(u/A^{1/2})+\log\frac{[A^{1/2}+(A-v^2)^{1/2}]}{v}=B\ (\text{constant})\dagger$$

is the finite equation of the geodesics.

Clearly this last equation may be written in the form

$$v=2ab\,e^{\sec^{-1}(u/a)}/(e^{2\sec^{-1}(u/a)}+b^2),$$

where a and b are constants, which, under the given transformation, becomes

$$x^2-2abxy+b^2=0.$$

No. 455. Proposed by *H. T. R. Aude*, Colgate University.

If an integer N is written in the scale r using place value, the symbol zero and the other necessary $r-1$ symbols, it will require at least one of these symbols (digits). Let the sum of these digits be S_1, which is expressed in fewer digits than N. Let S_2 be the sum of the digits of S_1, and so on for $S_3, S_4,\cdots$, until only one digit is needed. Denote the last number by $\sum$. Thus for 3457 in the scale of ten, $S_1=19$, $S_2=10$, $\sum=1$. For 3457 in the scale of eight, $S_1=23$, and $\sum=5$.

1. Show that in the scale of nine, the sum of any number divisible by four and the number preceding it is a number whose $\sum$ is seven.

2. Show that in the scale of seventeen, the sum of any four successive numbers, where the greatest is a multiple of four, is a number whose $\sum$ is ten.

†Eisenhart, *Differential Geometry*, page 217.

3. Show that in the scale of ten the sum of any three successive numbers, where the greatest is a multiple of three, is a number whose $\sum$ is six.

Solution by *M. I. Chernofsky*, Yeshiva College, New York City.

1. The sum of the numbers in question is $N=8k-1$. Thus $N\equiv 7 \pmod 8$. Since N is in the scale of nine,

$$N=9^p a_p+9^{p-1}a_{p-1}+\cdots+9a_1+a_0\equiv(7 \bmod 8)$$

becomes, since nine is congruent to unity (mod 8),

$$a_p+a_{p-1}+\cdots+a_1+a_0\equiv 7 \pmod 8.$$

Thus $S_1\equiv 7 \pmod 8$. By a finite number of repetitions of this argument, $S_2\equiv 7 \pmod 8$, $S_3\equiv 7 \pmod 8, \cdots, \sum=7$.

2. The sum of the numbers in question is $N=16k-6$, whence $N\equiv 10 \pmod{16}$. Since $17\equiv 1 \pmod{16}$, an argument analogous to that used above may be employed here to show that successively $S_1\equiv 10 \pmod{16}$, $S_2\equiv 10 \pmod{16}, \cdots, \sum=10$.

3. The sum of the given numbers in this case is $N=9k-3$, so that $N\equiv 6 \pmod 9$. As above, it may be shown that successively $S_1\equiv 6 \pmod 9$, $S_2\equiv 6 \pmod 9, \cdots, \sum=6$.

No. 457. Proposed by *N. A. Court*, University of Oklahoma.

The line joining the center of the second twelve point sphere of an orthocentric tetrahedron to the midpoint of the segment joining the orthocenter to a vertex passes through the corresponding vertex of the twin tetrahedron.

Solution by the *Proposer*.

The orthocenters A', B', C' of the faces DBC, DCA, DAB of the orthocentric tetrahedron $(T)=ABCD$ are the projections upon these faces of the orthocenter H of (T), hence A', B', C' lie on the sphere (DA) having the segment DH for diameter.

The orthocenters A', B', C' lie on the second twelve point sphere of (T) (see the proposer's Modern Pure Solid Geometry, p. 262, Art. 800. Macmillan, 1935), and also on a sphere passing through A, B, C, which sphere has for center the vertex D_0 of the twin tetrahedron (T_0) of (T) (*ibid*, p. 64, Art. 217).

The three spheres considered are thus coaxal, hence their centers are collinear, which proves the proposition.

PROPOSALS

No. 473. (Corrected). Proposed by *Howard D. Grossman*, New York City.

Prove the obvious generalization of the following relation:

$$\sum_{x=1}^{n} x^5 = \frac{n(n+1)}{6!} \begin{vmatrix} 2 & 0 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 & n \\ 1 & -3 & 4 & 0 & n^2 \\ -1 & 4 & -6 & 5 & n^3 \\ 1 & -5 & 10 & -10 & n^4 \end{vmatrix}$$

where the portion of the determinant below the principal diagonal is identical with a portion of the Pascal triangle except for the negative signs in alternate diagonals. The determinant is unchanged in value if all signs are made positive and n is replaced by $n+1$.

No. 479. Proposed by *F. A. Lewis*, University of Alabama.

Prove that the following "trisection" is valid for certain angles. With vertex 0 as the center, describe arc AB of any convenient radius r and trisect chord AB. Produce AB to C so that the distance from C to the first interior point of trisection equals r. With C as center and radius r describe the arc cutting arc AB at T. Then for certain angles, $\angle TOB = (\angle AOB)/3$.

No. 480. Proposed by *E. P. Starke*, Rutgers University.

Andy and Bill have five pennies each. These they match until one of them loses all. At each move the chances are even that Andy loses a penny to Bill or wins one from Bill. Find (a) the average number of moves required, (b) the most likely number of moves, and (c) the number, M, of moves such that the play is more likely to end on or before M moves than after, and is more likely to end on or after M moves than before.

No. 481. Proposed by *Nelson Robinson*, Norfolk, Virginia.

What is the locus of the center of a circle inscribed to a variable triangle of fixed base and height?

No. 482. Proposed by *Howard D. Grossman*, New York City.

It is known that a plane is divided by n lines into $(n^2+n+2)/2$ regions and that space is cut by n planes into $(n^3+5n+6)/6$ regions.

Prove that a plane is divided by n circles into n^2-n+2 regions, exactly twice as many as by $n-1$ lines; and that space is cut by n spheres into $(n^3-3n^2+8n)/3$ regions, exactly twice as many as by $n-1$ planes. Each of these numbers is the maximum possible and occurs only when each figure has distinct intersections with all the others.

No. 483. Proposed by *Paul D. Thomas*, Sherburne, N. Y.

Given two fixed non-concentric circles (A), (B). A third circle, (C), has a fixed radius but its center, C, moves on (B). Find the envelope of the radical axis of the circles (A), (C).

No. 484. Proposed by *H. C. D. McCluskey*, student, Colgate University.

A cylinder, without top, weighs m grams. Its center of gravity is assumed to be c centimeters below its center. To what depth must water be added to bring the center of gravity of the combined mass to its lowest possible point?

No. 485. Proposed by *Paul D. Thomas*, Sherburne, N. Y.

Prove that if perpendiculars are drawn from the feet of the altitudes of a triangle to the adjacent sides, the feet of these perpendiculars lie on a circle.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Calculus. By Alfred L. Nelson, Karl W. Folley, and William M. Borgman. D. C. Heath and Co., Boston. 1942. x+356 pages, \$2.75.

The authors state in their preface their concern for the technical student and this concern is apparent throughout their text. But this concern is exhibited by an increase, rather than a restriction, in the breadth of their treatment, and the *non-technical student* will find it all the more suitable for his use.

The 333 pages of text are intended to provide material for four class meetings per week throughout the year. The problems are in suitable abundance and variety. All the standard types to be found in most texts are included, along with a number of problems which, while taken from special fields, are quite intelligible to the non-technical student. The excellent figures are a striking feature. One notices the consistent use of the symbols D_x (with a passing reference to, and a word of justification for, the classical fractional notation) and, the chemist's preference, l_n.

Many of the elementary applications of derivatives and integrals are introduced and illustrated early in the course with simple algebraic expressions before the more general formulas are developed. Thus integration is first introduced in Chapter III on p. 37.

Occasional references are made to Dresden's *Calculus*, for proofs that are omitted: e. g., the existence of the limit defining *e*, and of the limit of the sum defining the definite integral. It is natural to contrast the two texts whose objectives are so different. Dresden seeks to provide at once the logical foundations and the models of logical rigor from which the student can proceed, with nothing to be unlearned or filled in. The purpose is admirable, but it is hard to see how justice can be done to Dresden's text in eight semester hours with any but a very exceptional class. The present authors, however, seek to develop calculus as a working tool, as fully and as firmly as time permits, leaving the finer points of logic to be investigated independently or learned in a later course by those students who are interested. One gets the impression that the authors have done very well.

The University of Chicago. A. S. HOUSEHOLDER.

Analytic Geometry. By H. P. Pettit and P. Luteyn. The Dryden Press, Inc., New York, 1942. x+196 pages. \$2.20.

If an instructor is searching for a text in plane and solid Analytic Geometry that contains the usual topics, though purposely brief and compact, then this book by Pettit and Luteyn should be a happy find. By turning through the book somewhat rapidly the general appearance is spontaneously pleasing, but a more critical review brings out the fact that the well chosen exercises seem crowded and the figures unnecessarily blurred. Furthermore, it is to be regretted that the publisher followed the general practice and did not see fit to place all figures together with their explanations so that they could be read without turning the page.

The authors have done a fine job of formulating a text that could be easily completed in a minimum of time. Yet they have not omitted any of the essential topics that are desirable in a first course in Analytic Geometry.

In order to utilize the unusually good summaries that are included at the end of most of the chapters, I would follow the suggestions of the authors and refer the better students to other books in the subject and have them select and classify appropriate exercises.

Illinois Institute of Technology. SAMUEL F. BIBB.

Essentials of Astronomy. By John Charles Duncan. Harper and Brothers, New York, 1942. 161+21 pp.; $1.85.

If the purpose in writing an elementary text book is to introduce the student to the subject and to inspire him to further acquaintance with it, rather than to impress him with its vast intricacies and its many technical ramifications, then the author has written a good text book on Astronomy.

He has definitely broken away from the traditional mode of writing a fact-laden compendium of a subject and calling it a textbook. This is shown both by the *many* and the often *important matters* which he hardly mentions. Few names or dates are given by the author.

The reader gets the impression that the author has given only the material which a good student might be expected to remember, say a year or two after taking the course. He senses an apparent yielding to the current sophomoric demand for "painless" education—no mathematics, no rigorous logic of methods to be mastered.

But the author is a scholar, solidly trained, continuously productive, a genial critic. He has packed his short sentences and paragraphs with fundamentals expressed in simple, direct style.

Fourteen appendices and excellent star maps conclude the book.

This reviewer stopped reading, to laugh, when on page 65 he came upon the geometric demonstration of the law of areas in a two-body system, usually reserved for introductory courses in celestial mechanics.

Essentials is a consistent, balanced, very modern textbook. The reviewer will use it for his favorite class this year.

Northwestern University. OLIVER J. LEE.

Spherical Trigonometry with Naval and Military Applications. By Lyman M. Kells, Willis F. Kern and James R. Bland. McGraw-Hill Book Company, Inc., New York and London, 1942. 163 pages; $1.50: with tables, 282 pages; $2.40.

This text comes from the publisher at an especially appropriate time. The fact that the authors are located at the United States Naval Academy lends interest to their treatment of this topic which was formerly much neglected at non-technical schools.

The titles of the chapters are: I, Logarithms; II, Review of Plane Trigonometry; III, The Right Spherical Triangle; IV, Applications; V, The Oblique Spherical Triangle; VI, Applications. Appendix A gives a discussion of the mil; B, the range finder; C, stereographic projections and graphical solution of spherical triangles; and D, vectors, relative movements and mooring board problems.

The discussion of logarithms (including exponential equations) in I is fairly complete. The standard formulas of plane trigonometry are collected together in II with

corresponding problems in military settings. In III the propositions of solid geometry that are needed are set forth. Napier's rules along with rules for quadrants are given. The ambiguous case is discussed. The possibility of no solution is mentioned in a problem. Polar and quadrantal triangles are discussed. The chapter closes with solution of oblique triangles by auxiliary right triangles. In IV the simpler aspects of navigation are considered. Formulas for plane sailing and middle latitude sailing are given. The Mercator Chart is discussed. In V we find first the law of sines and the laws of cosines for angles and sides. The half-angle formulas are developed but the half-side formulas are relegated to problems. Napier's analogies follow. A detailed discussion of the six cases is given. The chapter concludes with an alternative solution by the use of haversines of the important *two sides and included angle case.* In VI we find applications of the methods developed in the preceding chapter. Definitions of astronomical terms of use to the navigator are given. The standard problems such as finding the latitude, the local apparent time, sidereal time, etc., are treated in some detail with ample figures and many illustrative examples. Use of the Sumner line in determining a fix is discussed.

The printing is excellent and the figures have been prepared with care. Concise computational forms for the standard problems are given. Answers are included.

The reviewer feels that not only does the text give an excellent introduction to the subject but that even the terminology used in connection with military problems ought to aid considerably in furnishing the student with background for his later work.

Northwestern University. O. G. HARROLD.

Plane and Spherical Trigonometry. By Paul R. Rider. The Macmillan Book Company, New York, 1942. xiv+275 pages.

In preparing this book the author has arranged the material in such a way that the book might well be used as a text in almost any course in trigonometry. Schools offering the regular course in Plane and Spherical Trigonometry should be able to cover all of the topics in the book. If a shorter course is desired certain topics or chapters may well be omitted without interfering with the unity of the course, since the chapters are, to a large extent, independent of each other.

The chapter on the special definitions of the trigonometric functions and right triangles is discussed before the more general definitions. Those instructors who would prefer the general definitions first, could interchange the two chapters and present the course as it is presented in most texts.

There are numerous applications which should help stimulate the student's interest in the course. One feature of the book, which every instructor in a technological school should appreciate, is the fact that the author makes use of approximate numbers and emphasizes the question of significant figures. Too often instructors fail to point out this important topic to their students. The exercises throughout the book are numerous, arranged according to difficulty, and answers are given to the odd-numbered exercises; and numerous illustrations are worked out in detail to guide the student.

The discussion of the graphs of the trigonometric functions and of the inverse functions is good. The principal values of the inverse functions are emphasized and illustrated graphically for each function.

The book as a whole should prove to be very teachable from the standpoint of both the teacher and student.

North Carolina State College. H. V. PARK.

Basic College Mathematics. By Carl Wallace Munshower and James Fletcher Wardwell. Henry Holt and Company, New York, 1942. xi+612 pages; $3.20.

The first sentence of the preface says "This book surveys and unifies, through the concept of function, the material of the basic collegiate mathematics curriculum—algebra, trigonometry, analytic geometry, and the elements of calculus." Various attempts of this kind have been made in other texts with varied unifying concepts. The chapter headings will show how the authors attack the problem: 1, *introduction;* 2, *functions of first degree;* 3, *the power function;* 4, *the quadratic function;* 5, *summation;* 6, *polynomial functions;* 7, *other algebraic functions;* 8, *logarithmic functions;* 9, *exponential functions;* 10-14 (inclusive) *trigonometry;* 15, *polar coordinates;* 16, *complex numbers;* 17, *hyperbolic functions;* 18, *implicit functions of second degree;* 19, *implicit functions relating three variables; 20, determinants; 21, permutations, combinations, probability.* There follow answers to odd-numbered problems, tables of assorted kinds (pp. 519-593), twelve pages of formulas for reference, and an index (6 pages).

The printing of the book is very clear and attractive. On page 49 one digit is illegible, and the same is true of a word on page 53 and of a letter on page 493; but these are the only slips of the kind noted in the entire volume.

There is a rather surprising number of errors in the book, the presence of some of them being due to faulty proof-reading. The authors will, of course, go through it very carefully when the time comes for a second edition, and it would be an excellent idea to have a page or so of *Errata* printed and sent along with the present edition. Some of the mistakes seem serious enough to lead young students into wrong mathematical habits.

Problems are given from "business, economics, military science, navigation, psychology, and sociology," as well as chemistry and physics. It is to be hoped that problems involving, without explanation, such terms as entropy, parallax of a star, ohms, amperes, adiabatic expansion, index of refraction, and electromotive force will be given only to students acquainted with the subject in question. Otherwise, though in every case the mathematics of the problem can be done, the meaning of the result will be lost.

The book builds up from a study of the number system, and gives what may be called a foundation course. It is excellent on mathematical induction and on problems, especially in trigonometry. Treatment of equations like $\sin (x/2)=1/2$, or $\cos 3x=0$ is especially neat. On page 4 will be found a good fallacy, showing what division by zero will do. Differentiation and integration are carried along side by side, instead of having the first completed before the second is attempted. This has been very well done, and it seems to be a sound procedure. Enough is given on vectors so that students ought to feel quite familiar with that important subject. The normal form of the equation of a straight line is found first in polar coordinates and then transformed to rectangular coordinates. This is clever treatment that young students are sure to enjoy.

Almost anything anyone would want to teach in a mathematics course to college freshmen is somewhere in this book. Many teachers do not have a course meeting five times a week for a year, for which the text is planned; but the authors offer in the preface suggestions for choice for shorter courses. Most of the book is certainly held in line by the notion of function, and the reviewer feels that it gives a good survey course for students who plan to take no more mathematics.

Wellesley College. MARION E. STARK.